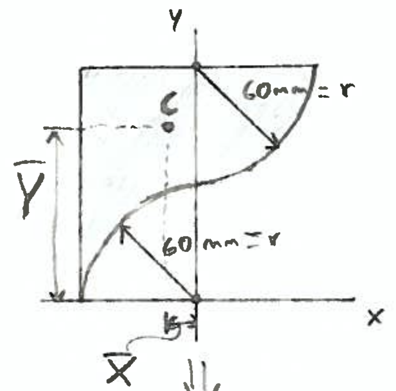
# CH 5

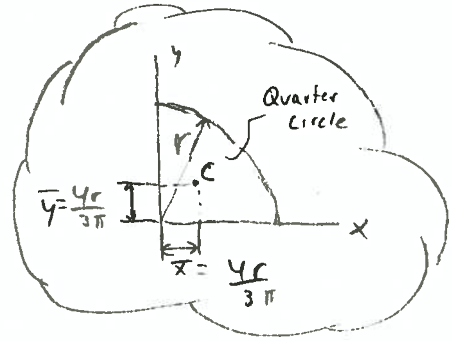
VECTOR MECHANICS FOR ENGINEERS - STATICS - 9TH EDITION

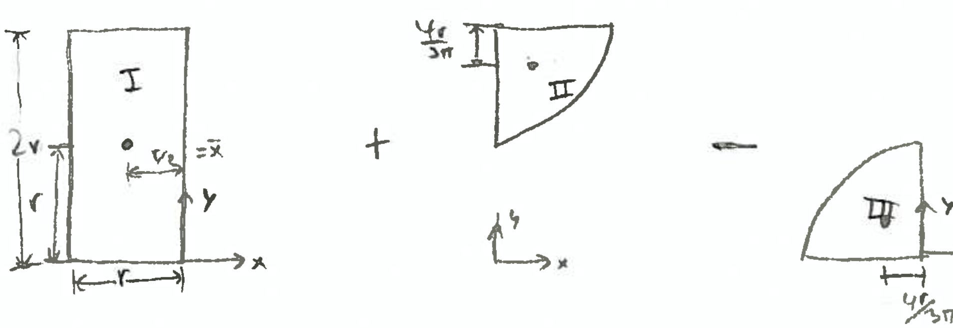
5.9) Finding the centroid of an area.

Given: The Following diagram

Find: The centroid of the area shown.




| Section | $A\ (mm^2)$ | $\bar{x}\ (mm)$ | $\bar{y}\ (mm)$ | $\bar{x}A\ (mm^3)$ | $\bar{y}A\ (mm^3)$ |
|---|---|---|---|---|---|
| I | $(r)(2r) = 2r^2$ | $-r/2$ | $r$ | $-r^3$ | $2r^3$ |
| II | $\frac{\pi r^2}{4}$ | $4r/3\pi$ | $2r - \frac{4r}{3\pi}$ | $\left(\frac{\pi r^2}{4}\right)\left(\frac{4r}{3\pi}\right) = \frac{r^3}{3}$ | $\frac{\pi r^2}{4}\left(2r - \frac{4r}{3\pi}\right)$ |
| III | $-\frac{\pi r^2}{4}$ | $-4r/3\pi$ | $4r/3\pi$ | $\left(\frac{-\pi r^2}{4}\right)\left(\frac{-4r}{3\pi}\right) = \frac{r^3}{3}$ | $-\frac{r^3}{3}$ |

$\Sigma A = 2r^2$

$\Sigma \bar{x}A = \frac{r^3}{3} + \frac{r^3}{3} - r^3$

$\Sigma \bar{x}A = -\frac{r^3}{3}$

$\Sigma \bar{y}A =$

$$\Sigma \bar{y}A = 2r^3 + \frac{\pi r^2}{4}\left(2r - \frac{4r}{3\pi}\right) - \frac{r^3}{3}$$

$$= r^3\left(2 + \frac{\pi}{2} - \frac{1}{3} - \frac{1}{3}\right) = r^3\left(\frac{4}{3} + \frac{\pi}{2}\right)$$

Now, $\bar{X}\Sigma A = \Sigma \bar{x}A$, $\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{-\frac{r^{\cancel{3}}}{3}}{2\cancel{r^2}} = \frac{-r}{6} = \frac{-\cancel{60}^{10}\,mm}{\cancel{6}} = \boxed{-10.00\,mm = \bar{X}}$

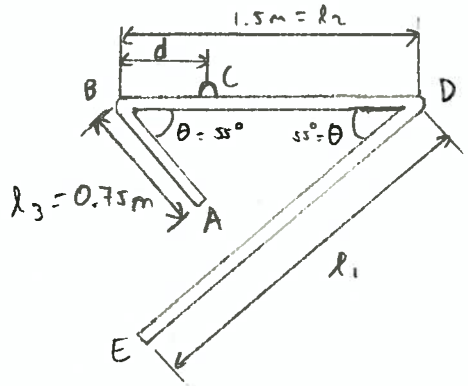
And, $\bar{Y}\Sigma A = \Sigma \bar{y}A$, $\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{r^{\cancel{3}}\left(\frac{4}{3} + \frac{\pi}{2}\right)}{2\cancel{r^2}} = r\left(\frac{2}{3} + \frac{\pi}{4}\right) = (60mm)\left(\frac{2}{3} + \frac{\pi}{4}\right)$

$\boxed{\bar{Y} = 87.1\,mm}$

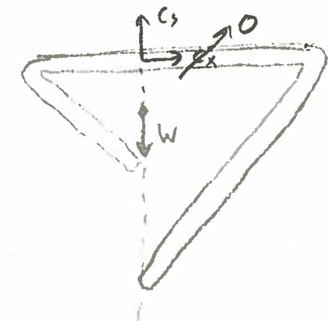
5.29) Hanging aluminum tubing.

Given: • Member ABCDE is a single piece of aluminum tubing
• member is supported at C.
• $\ell_1 = 2\,m$ , $\theta = 55°$

Find: d such that portion BCD is horizontal

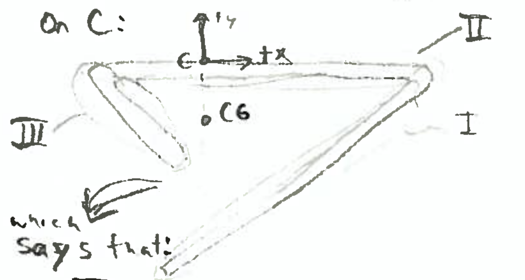



FBD of member ABCDE




In order to maintain equib the Weight must be on the line of action of $C_y$

So if we set our coordinate axes on C:

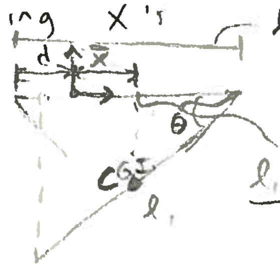



which says that:

$\overline{X} = 0$

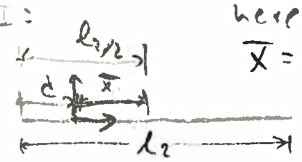
Finding $\overline{x}$'s

I:



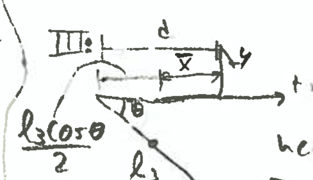

here, $\overline{x} = \ell_2 - d - \frac{\ell_1\cos\theta}{2}$

II:




here $\overline{x} = \frac{\ell_2}{2} - d$

III:




here, $\overline{x} = \frac{\ell_3\cos\theta}{2} - d$

| Section | L(m) | $\overline{x}$(m) | $\overline{x}L$ ($m^2$) |
|---|---|---|---|
| I | $\ell_1$ | $\ell_2 - d - \frac{\ell_1\cos\theta}{2}$ | $\ell_1\ell_2 - \ell_1 d - \frac{\ell_1^2\cos\theta}{2}$ |
| II | $\ell_2$ | $\frac{\ell_2}{2} - d$ | $\frac{\ell_2^2}{2} - \ell_2 d$ |
| III | $\ell_3$ | $\frac{\ell_3\cos\theta}{2} - d$ | $\frac{\ell_3^2\cos\theta}{2} - \ell_3 d$ |

$\Sigma L = \ell_1 + \ell_2 + \ell_3$

$$\overline{X}\,\Sigma L = \Sigma\overline{x}L \;,\; \overline{X} = \frac{\Sigma\overline{x}L}{\Sigma L} \;,\; \text{and } \overline{X} = 0 \;\text{ so, }\; \frac{\Sigma\overline{x}L}{\Sigma L} = 0$$

$$\frac{\left(\ell_1\ell_2 - \ell_1 d - \frac{\ell_1^2\cos\theta}{2}\right) + \left(\frac{\ell_2^2}{2} - \ell_2 d\right) + \left(\frac{\ell_3^2\cos\theta}{2} - \ell_3 d\right)}{\cancel{(\ell_1 + \ell_2 + \ell_3)}} = 0$$

$$\ell_1 d + \ell_2 d + \ell_3 d = \ell_1\ell_2 - \frac{\ell_1^2\cos\theta}{2} + \frac{\ell_2^2}{2} + \frac{\ell_3^2\cos\theta}{2}$$

5.29 Part II)

$$d(l_1+l_2+l_3) = \frac{2l_1l_2 - l_1^2\cos\theta + l_2^2 + l_3^2\cos\theta}{2}$$

$$d = \frac{2l_1l_2 - l_1^2\cos\theta + l_2^2 + l_3^2\cos\theta}{2(l_1+l_2+l_3)}$$

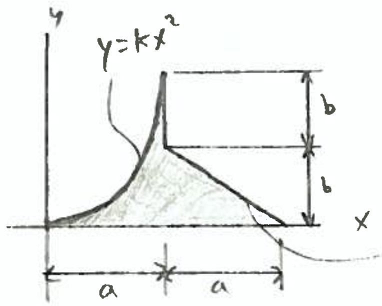
$$d = \frac{2(2m)(1.5m) - (2m)^2\cos 55^\circ + (1.5m)^2 + (0.75m)^2\cos 55^\circ}{2(2m + 1.5m + 0.75m)}$$

$$\boxed{d = 0.739m}$$

5.44) Finding the centroid of an area by integration.

Given: The following graph.

Find: the centroid of the area shown in terms of a and b.




we can use the following formulas:

$$\bar{x}A = \int x\,dA \quad \text{and} \quad \bar{y}A = \int y\,dA$$

This curve is a line $\rightarrow y = mx + C$, where, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - b}{2a - a} = \frac{-b}{a}$

also, when $y = 0$, $x = 2a$

So we have $0 = 2\cancel{a}\left(\frac{-b}{\cancel{a}}\right) + C$

$C = 2b$

First, we need to find $k$ in $y = kx^2$

we see that one of the points on the curve $y = kx^2$ is $(a, 2b)$

So, $(2b) = k(a)^2$, or $k = \frac{2b}{a^2}$

So, the first curve is: $y = \frac{2b}{a^2}x^2$

and the second curve was found to be $y = \frac{-b}{a}x + 2b$

The area is

$$A = \int_0^a \frac{2b}{a^2}x^2\,dx + \int_a^{2a}\left(\frac{-b}{a}x + 2b\right)dx$$

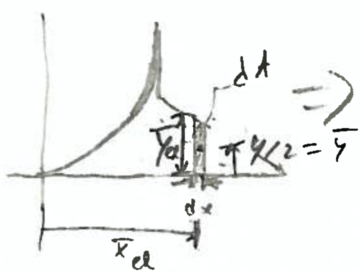
$$A = \frac{2b}{a^2}\left[\frac{x^3}{3}\right]_0^a + b\left[\frac{-x^2}{2a} + 2x\right]_a^{2a}$$

$$A = \frac{2b}{\cancel{a^2}}\left(\frac{a^{\cancel{3}}}{3} - 0\right) + b\left[\frac{-(2a)^2}{\cancel{2a}} + 2(2a) - \left(\frac{-a^{\cancel{2}}}{2\cancel{a}} + 2a\right)\right]$$

$$A = \frac{2ba}{3} + b\left[-\cancel{2a} + \cancel{4a} + \frac{a}{2} - \cancel{2a}\right]$$

$$A = \frac{2ba}{3} + \frac{ba}{2}, \quad A = \frac{4ba + 3ba}{6}, \quad A = \frac{7ba}{6}$$

Looking at the graph




$\Rightarrow$ So, $dA = y\,dx$

So $\bar{x}_{el} = x$

and $\bar{y}_{el} = \frac{y}{2}$

5.44 Part II)

Now we can find $\bar{x}$ and $\bar{y}$

$$\bar{x}A = \int \tilde{x}_{el}\,dA = \int xy\,dx$$

$$\bar{x}A = \int_0^a x\left(\frac{2b}{a^2}x^2\right)dx + \int_a^{2a} x\left(-\frac{b}{a}x + 2b\right)dx$$

$$\bar{x}A = \frac{2b}{a^2}\int_0^a x^3\,dx + b\int_a^{2a}\left(\frac{-x^2}{a} + 2x\right)dx$$

$$\bar{x}A = \frac{\cancel{2}b}{a^2}\left[\frac{x^4}{\cancel{4}\,2}\right]_0^a + b\left[\frac{-x^3}{3a} + \frac{\cancel{2}x^2}{\cancel{2}}\right]_a^{2a}$$

$$\bar{x}A = \frac{b}{\cancel{a^2}}\left[\frac{a^{\cancel{4}\,2}}{2} - 0\right] + b\left[\frac{-(2a)^3}{3a} + (2a)^2 - \left(\frac{-(a)^{\cancel{3}\,2}}{3\cancel{a}} + a^2\right)\right]$$

$$\bar{x}A = \frac{ba^2}{2} + b\left[\frac{-8a^{\cancel{3}\,2}}{3\cancel{a}} + 4a^2 + \frac{a^2}{3} - a^2\right]$$

$$\bar{x}A = \frac{ba^2}{2} + ba^2\left(\frac{-8+12+1-3}{3}\right)$$

$$\bar{x}A = \frac{ba^2}{2} + \frac{2ba^2}{3} = \frac{3ba^2 + 4ba^2}{6} = \frac{7ba^2}{6}$$

$$\bar{x} = \frac{1}{A}\left(\frac{7ba^2}{6}\right) = \frac{1}{\left(\frac{\cancel{7ba}}{\cancel{6}}\right)}\left(\frac{\cancel{7ba}^{\cancel{2}}}{\cancel{6}}\right) = \boxed{a = \bar{x}}$$

$$\bar{y}A = \int \bar{y}_{el}\,dA = \int\left(\frac{y}{2}\right)y\,dx = \int \frac{y^2}{2}dx, \quad \text{so,} \quad 2\bar{y}A = \int y^2\,dx$$

$$2\bar{y}A = \int_0^a\left(\frac{2b}{a^2}x^2\right)^2dx + \int_a^{2a}\left(-\frac{b}{a}x + 2b\right)^2dx$$

$$2\bar{y}A = \frac{4b^2}{a^4}\int_0^a x^4\,dx + b^2\int_a^{2a}\left(\frac{x^2}{a^2} - \frac{4x}{a} + 4\right)dx$$

$$2\bar{y}A = \frac{4b^2}{a^4}\left[\frac{x^5}{5}\right]_0^a + b^2\left[\frac{x^3}{3a^2} - \frac{2x^2}{a} + 4x\right]_a^{2a}$$

$$2\bar{y}A = \frac{4b^2}{\cancel{a^4}}\left[\frac{a^{\cancel{5}}}{5} - 0\right] + b^2\left[\frac{(2a)^3}{3a^2} - \frac{2(2a)^2}{a} + 4(2a) - \left(\frac{a^{\cancel{3}}}{3a^{\cancel{2}}} - \frac{2a^{\cancel{2}}}{\cancel{a}} + 4a\right)\right]$$

$$2\bar{y}A = \frac{4b^2a}{5} + b^2\left[\frac{8a^{\cancel{3}}}{3a^{\cancel{2}}} - \cancel{\frac{8a^2}{a}} + \cancel{8a} - \frac{a}{3} + 2a - 4a\right]$$

$$2\bar{y}A = \frac{4b^2a}{5} + b^2a\left(\frac{8-1-6}{3}\right) = \frac{4b^2a}{5} + \frac{b^2a}{3} = \frac{12b^2a + 5b^2a}{15}$$

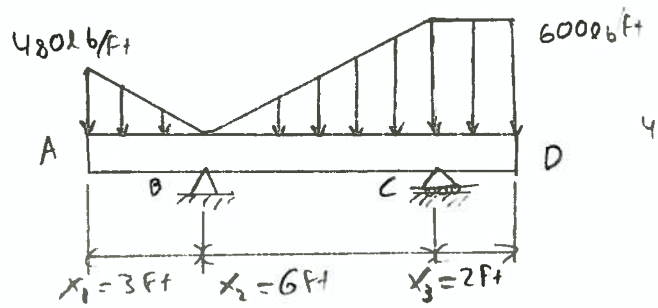
$$2\bar{y} = \frac{1}{A}\,\frac{17b^2a}{15} = \frac{1}{\left(\frac{7ba}{6}\right)}\left(\frac{17b^{\cancel{2}}\cancel{a}}{15}\right) = \frac{102}{105}b$$

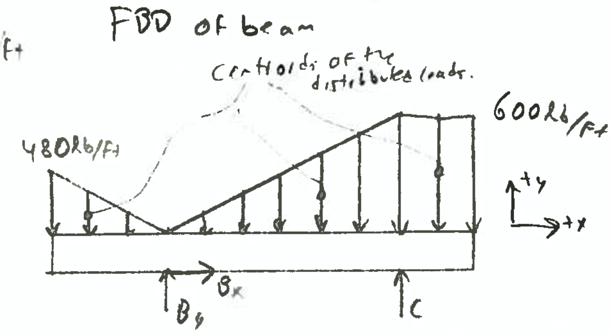
$$2\bar{y} = \frac{34}{35}b, \quad \text{So} \quad \boxed{\bar{y} = \frac{17}{35}b}$$

5.69) A loaded beam

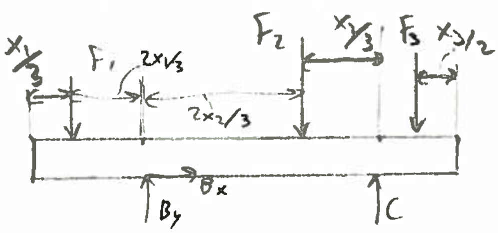
Given: the Following diagram

Find: The rxns (reactions) at the beam supports




We can replace the distributed forces with concentrated forces at the centroids of the the loads → The magnitude of these Concentrated Forces is the area under these respective load curves.




$\stackrel{+}{\rightarrow} \Sigma F_x = 0 \Rightarrow B_x = 0$

$+\uparrow \Sigma F_y = 0 \Rightarrow B_y + C - F_1 - F_2 - F_3 = 0$ ✱

$$\circlearrowleft + \Sigma M_B = 0 \Rightarrow F_1\left(\frac{2x_1}{3}\right) - F_2\left(\frac{2x_2}{3}\right) - F_3\left(x_2 + \frac{x_3}{2}\right) + x_2 C = 0$$

$$C = \frac{\frac{2F_2 x_2}{3} + F_3 x_2 + \frac{F_3 x_3}{2} - \frac{2F_1 x_1}{3}}{x_2}$$

$$C = \frac{4F_2 x_2 + F_3(6x_2 + 3x_3) - 4F_1 x_1}{6x_2}$$

$$C = \frac{4[(6ft)(600lb/ft)/2](6ft) + [600lb/ft)(2ft)](6(6ft) + 3(2ft)) - 4[(3ft)(480lb/ft)/2](3ft)}{6(6ft)}$$

$\boxed{C = 2360 lb \uparrow}$

Using eqn ✱

$B_y = F_1 + F_2 + F_3 - C$

$$B_y = \frac{(480\frac{lb}{ft})(3ft)}{2} + \frac{(600\frac{lb}{ft})(6ft)}{2} + (600\frac{lb}{ft})(2ft) - 2360 lb$$

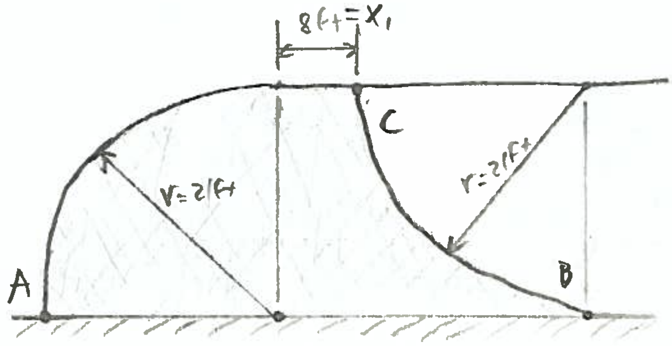
$B_y = \boxed{B = 1360 lb \uparrow}$

5.81) A concrete dam

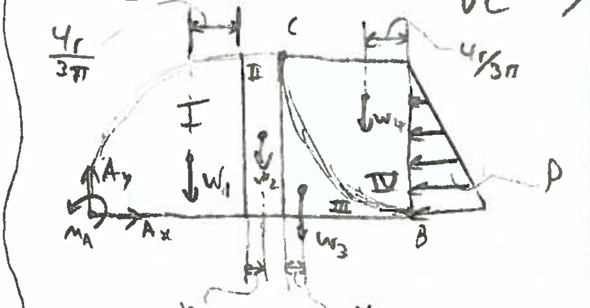
**Given:** width = 1 ft
$\gamma_{water} = 62.4\ lb/ft^3$
$\gamma_{concrete} = 150\ lb/ft^3$

Note:
$\gamma = \rho g$
$\gamma$ = (density)(gravity)

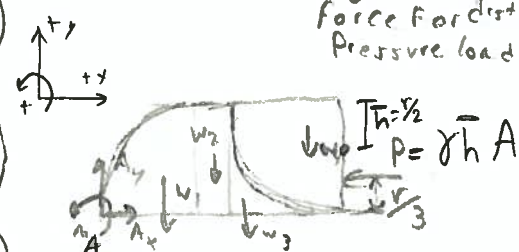
**Find:** (a) the resultant of the reaction forces exerted by the ground on the face AB of the dam
(b) The point of application of the resultant.
(c) The resultant of the pressure forces exerted by the water on the face BC of the dam



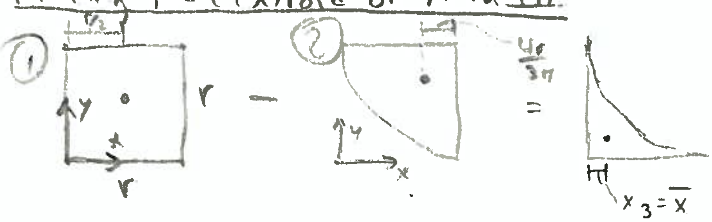

**FBD** of (dam + water above BC)




⇓ using concentrated force for distributed pressure load




Finding the centroid of Area III




| Section | $A$ | $\bar{x}$ | $\bar{x}A$ |
|---|---|---|---|
| 1 | $r^2$ | $r/2$ | $r^3/2$ |
| 2 | $\frac{\pi r^2}{4}$ | $r - \frac{4r}{3\pi}$ | $\frac{\pi r^3}{4} - \frac{r^3}{3}$ |

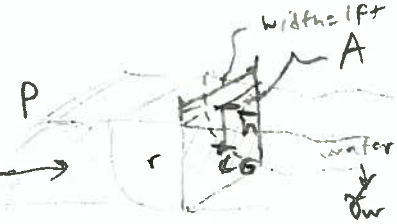
$$\bar{X} = \frac{\Sigma \bar{x} A}{\Sigma A} = \frac{\frac{r^3}{2} + \left(\frac{\pi r^3}{4} - \frac{r^3}{3}\right)}{r^2 + \frac{\pi r^2}{4}} = \frac{\frac{6r}{12} + \frac{3\pi r}{12} - \frac{4r}{12}}{\frac{4}{4} + \frac{\pi}{4}} = \frac{\frac{2r + 3\pi r}{\cancel{12}\,3}}{\frac{4+\pi}{\cancel{4}}} = \frac{2r + 3\pi r}{3(4+\pi)}$$

$$\bar{X} = X_3 = \frac{r(2+3\pi)}{(12+3\pi)}$$

**Now, Sum forces to find rxns:**

$+\rightarrow \Sigma F_x = 0 \Rightarrow \quad A_x - P = 0, \quad A_x = P$

The formula for the pressure force is: $\quad P = \gamma \bar{h} A$




where:
- $\gamma$ is the specific weight of the liquid (in this case water)
- $\bar{h}$ is the vertical distance from the surface of the water to the centroid of the surface the pressure is being applied to.
- A is the Area of the surface the pressure is being applied to (In this case its a rectangle of dimensions r × width)

5.81 Part II)

So,

$P = \gamma \bar{h} A$

is:

$P = \gamma_w \left(\frac{r}{2}\right)(w\, r) = \frac{\gamma_w r^2 w}{2}$ width

$P = \frac{(62.4\ lb/ft^3)(21ft)^2(1ft)}{2} = 13759.2\ lb$

$A_x = 13760\ lb$ (sig figs)

Note:
$w = mg$
$w = (\rho V)g$
$w = \rho g V$
$w = \gamma V$

Now in the y direction:

$+\uparrow \Sigma F_y = 0 \Rightarrow A_y - w_1 - w_2 - w_3 - w_4 = 0$

$A_y = w_1 + w_2 + w_3 + w_4$

where,

$w_1 = \gamma_c V_1 = (150\ lb/ft^3)\left(\frac{\pi (21ft)^2}{4}\right)(1ft) = 51954\ lb$

$w_2 = \gamma_c V_2 = (150\ lb/ft^3)(21ft)(8ft)(1ft) = 25200\ lb$

$w_3 = \gamma_c V_3 = \gamma_c (A_{square} - A_{quarter\ circle})\,\text{width}$

$= (150\ lb/ft^3)\left((21ft)^2 - \frac{\pi (21ft)^2}{4}\right)(1ft) = 14196\ lb$

$w_4 = \gamma_w V_4 = (62.4\ lb/ft^3)\left(\frac{\pi (21ft)^2}{4}\right)(1ft) = 21613\ lb$

So, $A_y = 51954\ lb + 25200\ lb + 14196\ lb + 21613\ lb$

$A_y = 112963\ lb$, $A_y = 113000\ lb$ (sig figs)

(a) The resultant of the rxn forces exerted by the ground on the face AB of the dam is simply:

$\bar{R} = A_x \hat{i} + A_y \hat{j}$

or $\bar{R} = 13760\ lb\ \hat{i} + 113000\ lb\ \hat{j}$

or $\boxed{\bar{R} = 13.76\ kips\ \hat{i} + 113.0\ kips\ \hat{j}}$

5.81 Part III)

(b) In order to find the point of application of $\bar{R}$ we need to sum moments about A, find $M_A$ and move $\bar{R}$ a distance x to the right in order to account for $M_A$

Sum of moments:

$$\curvearrowleft_{+} \Sigma M_A = 0 \Rightarrow M_A + P\left(\frac{r}{3}\right) - W_1\left(r - \frac{4r}{3\pi}\right) - W_2\left(r + \frac{x_1}{2}\right) - W_3(r + x_1 + x_3) - W_4\left(r + x_1 + r - \frac{4r}{3\pi}\right) = 0$$

$$M_A = W_1\left(r - \frac{4r}{3\pi}\right) + W_2\left(r + \frac{x_1}{2}\right) + W_3(r + x_1 + x_3) + W_4\left(2r + x_1 - \frac{4r}{3\pi}\right) - P\left(\frac{r}{3}\right)$$

$$M_A = (51954\,lb)\left(21ft - \frac{4(21ft)}{3\pi}\right) + (252000\,lb)\left(21ft + \frac{8ft}{2}\right) + (141196\,lb)\left(21ft + 8ft + \frac{(21ft)(2+3\pi)}{(12+3\pi)}\right)$$
$$+ (21613\,lb)\left(2(21ft) + 8ft - \frac{4(21ft)}{3\pi}\right) - (13759\,lb)\left(\frac{21ft}{3}\right)$$

$M_A = 2519550.672$ lb ft

In order to get rid of $M_A$ we move R, x to the right
So we have:

$$M_A = R_y x$$
(Rx does not contribute to the moment so only $R_y$ is needed)

Or $M_A = A_y x$

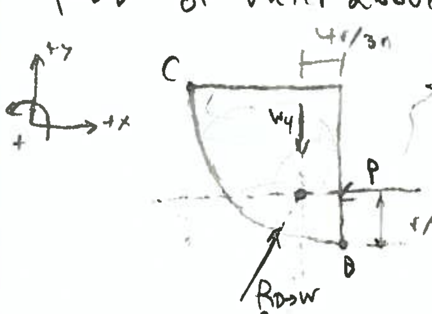
$$x = \frac{M_A}{A_y} = \frac{2519550.672\,\cancel{lb}\,ft}{112963\,\cancel{lb}}$$

$x = 22.3$ ft

So, the point of application is $\boxed{22.3ft \text{ to the right of } A}$

(c) Resultant of water Forces ($R_{w \to D}$)
resultant of pressure forces by water on dam

FBD of water above BC:




3 forces: in order to have equib. forces must be concurrent

Resultant of pressure forces by dam on water

$$\xrightarrow{+} \Sigma F_x = 0 \Rightarrow (R_{D \to W})_x - P = 0$$

$(R_{D \to W})_x = P$

$(R_{D \to W})_x = 13760\,lb$

$$+\uparrow \Sigma F_y = 0 \Rightarrow (R_{D \to W})_y - W_4 = 0$$

$(R_{D \to W})_y = W_4$

$(R_{D \to W})_y = 21600\,lb$

Or, $R_{w \to D} = 25.6$ kips, 57.5°

So $\bar{R}_{D \to W} = 13760\,lb\,\hat{i} + 21600\,lb\,\hat{j}$

We want $R_{W \to D}$
since $R_{W \to D}$ and $R_{D \to W}$ are equal and opposite $\Rightarrow$ $\boxed{\bar{R}_{W \to D} = -13760\,lb\,\hat{i} - 21600\,lb\,\hat{j}}$